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some given assigned arrangement. This result can be used for determining statistically whether stock judging is more than a matter of chance.

O. S. ADAMS, *Secretary-Treasurer*.

ON CERTAIN PROPERTIES OF MAKEHAM'S LAWS OF MORTALITY WITH APPLICATIONS.¹

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1. Introduction. Early in the development of the mathematics of life insurance, the calculations became laborious. This situation led to efforts to discover laws of mortality from which monetary values for life insurance could be calculated. As early as 1725, De Moivre in his *Annuities upon Lives*, put forth his famous hypothesis that the decrements d_x of the number living l_x at age x in a mortality table is constant for all values of x . Very little data were available to verify this hypothesis, but even the data of that time should have made it appear that this assumption could at most be a rough approximation. The De Moivre hypothesis is, of course, equivalent to a statement that the number living l_x at age x can be given by a linear function of x .

Just about one hundred years later in the *Philosophical Transactions* of 1825,² Gompertz published the so called law of mortality which bears his name. Before discussing the hypothesis of Gompertz, it is desirable to explain the meaning of the expression "force of mortality." This expression is in common use by actuaries at the present time in connection with the doctrines of Gompertz and in other connections. It should perhaps be stated that Gompertz did not use the expression "force of mortality" in his papers, but used instead the expression "intensity of mortality" which has no technical meaning at present.

The "force of mortality" μ_x at any age x may be defined as the rate of change of the average number living l_x of an indefinitely large class of persons of age in the neighborhood of x , per individual of the group l_x . That is to say, if the function l_x has a derivative with regard to x ,

$$\mu_x = -\frac{1}{l_x} \frac{dl_x}{dx},$$

the negative sign being used so that μ_x is positive.

In other words, we conceive a continuous function l_x which is proportional to the average number living at age x , out of an indefinitely large group of lives of age in the neighborhood of x . Then the force of mortality is defined as the derivative of this function with regard to age or time divided by l_x . Obviously

¹ Read under slightly different title before the Mathematical Association of America, September 6, 1920.

² "On the nature of the function expressive of the law of human mortality and a new mode of determining the value of life contingencies," pp. 513-583.

l_x is a decreasing function. The negative sign is placed before the derivative dl_x/dx so that μ_x is positive. Gompertz gave a rather wordy discussion of his hypothesis showing its reasonableness.

Briefly, his hypothesis may be stated by saying that the rate of change of the force of mortality at any time is proportional to the force of mortality. That is,

$$\frac{d\mu_x}{dx} = c'\mu_x,$$

or

$$\mu_x = Bc^x, \quad (1)$$

where B and c are parameters. Then from (1) and the definition of force of mortality, we have

$$-\frac{1}{l_x} \frac{dl_x}{dx} = Bc^x. \quad (2)$$

Integrating, we have in the usual form of Gompertz's law,

$$l_x = kg^{c^x}, \quad (3)$$

where k , g , and c are the parameters.

It is of some interest to know the view of Gompertz in regard to the real character of his derivation after he gave such a lengthy discussion of the reasonableness of his hypotheses. He remarked that "this equation between the number living and the age becomes deserving of attention not in consequence of its hypothetical deduction, which in fact is congruous with many natural effects, as for instance the exhaustion of the receiver of an air pump by strokes repeated at equal intervals of time, but it is deserving of attention because it appears corroborated during a long portion of life by experience." In the paper of 1825, Gompertz applied his function to portions of the Northampton Tables, Carlisle Tables, Desparcieux Tables, and certain Swedish Tables.

In the *Philosophical Magazine* for 1839,¹ De Morgan showed that if the law of Gompertz applies to the whole of life, then Simpson's Rule, which has often been used as an approximation, to simplify joint life annuity calculations, would be rigidly correct.

In 1860, Makeham suggested² his first modification of the law of Gompertz and in 1865 published a valuable paper setting forth the development of his suggestion. The hypothesis back of Makeham's first modification is that the force of mortality is given by

$$\mu_x = A + Bc^x. \quad (4)$$

That is, the force of mortality consists of two parts—the one part increases in geometrical progression with age and the other is a constant. The addition of the constant seems reasonable when we recall that for certain diseases and accidents, the tendency to death seems practically independent of age.

¹ "On the rule for finding the value of an annuity on three lives," vol. 15, p. 337.

² *Journal of the Institute of Actuaries*, vol. 8, 1860, pp. 301-310; vol. 13, 1867, pp. 325-358.

Then from (4) and the definition of force of mortality, we have

$$-\frac{1}{l_x} \frac{dl_x}{dx} = A + Bc^x, \quad (5)$$

and by integration we have for Makeham's first modification

$$l_x = ks^x g^{c^x}, \quad (6)$$

where k , s , g , and c are the parameters.

The determination of these parameters to fit a given table of mortality under some criterion such as that of least squares is a problem of interest in attempts at accuracy and economy of time. This problem has received considerable attention in recent literature. It is perhaps sufficient for our purpose to state that the main methods used are:

- (a) Method of averaging results obtained from using four values of l_x to correspond to four assigned values of x .
- (b) Method of moments.
- (c) Method of least squares.

It may be of interest to write down the values of the parameters for some well-known table. For the American Experience Table, Arthur Hunter found the following:

$$\log_{10} g = -.00013205$$

$$\log_{10} c = .04579609$$

$$\log_{10} s = -.003296862$$

$$\log_{10} k = 5.03370116$$

With respect to fitting tables well with the Makeham function $l_x = ks^x g^{c^x}$, there are many examples of tables that have been fitted well enough for practical purposes. As a striking example of closeness of fit, I wish to refer to a graduation of a rural life population table¹ for the registration area of the United States by C. H. Forsyth and the writer. The resulting curve crossed the curve of the given data eleven times in the range from age 19 to 85 which we treated.

2. The advantages of a Makehamized table for joint life and survivorship insurances. It is the main purpose of this paper to exhibit certain features of the application of Makeham's function in the treatment of joint life and survivorship insurances. It is fundamental in joint life and in survivorship insurances and annuities that we be able to find in the simplest way possible the value of joint life annuities to two or more persons of any assigned ages that may arise in the applications. We wish to show how it becomes possible with a Makehamized table to replace a cumbersome and laborious method of finding joint life annuities by a simple and elegant plan.

¹ *Record of the American Institute of Actuaries*, June, 1911, p. 19.

The laborious character of joint life calculations, when a table is not Makehamized, should perhaps be indicated first. This can be shown with sufficient clearness by an illustration with only two lives of ages x and y . The joint life annuity for (x) ¹ and (y) is given by

$$a_{xy} = \frac{vl_{x+1}l_{y+1} + v^2l_{x+2}l_{y+2} + v^3l_{x+3}l_{y+3} + \cdots}{l_xl_y}, \quad (7)$$

where a symbol l with a subscript means the number living in the mortality table at the age indicated by the subscript, and v is the discount factor. That is,

$$v = \frac{1}{1+i},$$

where i is the rate of interest.

The formula (7) can be extended in an obvious manner to include three or more lives. For only two lives, in De Morgan's forms, we introduce commutation symbols as follows:

$$D_{xy} = l_xl_yv^{(x+y)/2},$$

$$C_{xy} = (l_xl_y - l_{x+1}l_{y+1})v^{(x+y+2)/2},$$

$$= vD_{xy} - D_{x+1:y+1}.$$

$$N_{xy} = D_{xy} + D_{x+1:y+1} + D_{x+2:y+2} + \cdots,$$

$$M_{xy} = C_{xy} + C_{x+1:y+1} + C_{x+2:y+2} + \cdots,$$

$$S_{xy} = N_{xy} + N_{x+1:y+1} + N_{x+2:y+2} + \cdots,$$

$$R_{xy} = M_{xy} + M_{x+1:y+1} + M_{x+2:y+2} + \cdots.$$

To exhibit the tabular values for any one of this set of symbols, say for D_{xy} , for all combinations of ages from 10 to 95 inclusive would require a double entry table with $(86)^2$ tabular entries if the rates of mortality of the lives of (x) and (y) are given by different mortality tables, as in the case of a husband and wife when we use McClintock's Annuitants' Tables² (Male and female). If the rates of mortality of x and y are given by the same mortality table, the $(86)^2$ values would involve duplicates since $D_{xy} = D_{yx}$, and the number of necessary tabular entries would be $_{87}C_2$.

For three lives of (x) , (y) , and (z) with rates of mortality from different mortality tables, there would be required $(86)^3$ tabular entries to give D_{xyz} . If the mortality rates for (x) , (y) , and (z) were given by a single mortality table, the number of necessary tabular entries would be $_{88}C_3$.

It becomes clear, without carrying the illustration to a larger number of lives, that, unless a very extensive joint life business were to be done, it would be impractical to prepare such tables.

¹ A symbol (x) is an abbreviation for "a person aged x ."

² M. M. Dawson, *Practical Lessons in Actuarial Science*, New York, vol. 2, 1905, pp. 315-325.

We should ordinarily find it more practical to treat each separate case by a method of approximation which consists in finding the approximate sum of the series of terms in (7). The method of approximation very often adopted consists in applying formulas derived from the Euler-Maclaurin formula of finite differences. Even with these methods of approximation, this plan is generally somewhat laborious.

3. Lives of unequal ages can be replaced by lives of equal ages. We shall now attempt to give a brief exposition of the method that can be substituted for the rather laborious method of finding joint life annuities, when the table of mortality is a Makehamized table.

In the usual notation, let

$${}_t p_{x_1 x_2 \dots x_m}$$

be the probability that each of m persons $(x_1), (x_2), (x_m)$ will live t years. Under Makeham's law, $l_x = k s^x g^{c^x}$ for all values of x in a certain interval. Then we have

$$\begin{aligned} {}_t p_{x_1} &= s^t g^{c^{x_1}(c^t-1)}, \\ {}_t p_{x_2} &= s^t g^{c^{x_2}(c^t-1)}, \\ &\vdots \\ {}_t p_{x_m} &= s^t g^{c^{x_m}(c^t-1)}. \end{aligned}$$

Then the value of the joint life annuity to $(x_1), (x_2), \dots, (x_m)$ is

$$a_{x_1 x_2 \dots x_m} = v p_{x_1 x_2 \dots x_m} + v^2 {}_2 p_{x_1 x_2 \dots x_m} + v^3 {}_3 p_{x_1 x_2 \dots x_m} + \dots + v^t {}_t p_{x_1 x_2 \dots x_m} + \dots, \quad (8)$$

$$= \sum_{t=1}^{t=\omega} v^t p_{x_1 x_2 \dots x_m} \dots, \quad (9)$$

$$= \sum_{t=1}^{t=\omega} s^m v^t g^{(c^{x_1} + c^{x_2} + \dots + c^{x_m})(c^t-1)}, \quad (10)$$

where ω is the highest age in the mortality table.

Let ξ be the equal age of m persons that may, for the purpose of finding the value of the annuity replace ages $x_1, x_2, \dots, x_m$.

Then ${}_t p_{\xi} = s^t g^{c^{\xi}(c^t-1)}$ and the probability that m persons each of age ξ survive t years is

$$({}_t p_{\xi})^m = s^{mt} g^{mc^{\xi}(c^t-1)}. \quad (11)$$

If we determine ξ so that

$$mc^{\xi} = c^{x_1} + c^{x_2} + \dots + c^{x_m}, \quad (12)$$

we may clearly write in place of (10),

$$a_{x_1 x_2 \dots x_m} = \sum_{t=1}^{t=\omega} v^t ({}_t p_{\xi})^m = s^{mt} g^{mc^{\xi}(c^t-1)}. \quad (13)$$

The solution of (12) for ξ is generally obtained by the use of a table of forces of mortality. This plan is easily shown as follows:

From $l_x = ks^xg^{c^x}$, we have

$$\mu_x = -\frac{1}{l_x} \frac{dl_x}{dx} = -\log s - c^x \log g \log c, \quad (14)$$

From (12) and (14), we have that

$$m\mu_\xi = \mu_{x_1} + \mu_{x_2} + \cdots + \mu_{x_m}. \quad (15)$$

Then the value of ξ that satisfies (15) also satisfies (12). As tables of forces of mortality usually accompany a Makehamized mortality table, we solve for ξ by interpolation from such tables. With m equal lives substituted for the m unequal lives, there would be required for tabular entries of a commutation symbol, say for a $D_{\xi\xi} \cdots$, to cover all ages from some age such as 10 to an age such as 95 inclusive, simply 86 values instead of the impractically large number of values mentioned above.

4. Law of uniform seniority. For simplicity, let us consider first two lives (x_1) and (x_2). The equivalent equal age ξ is given by

$$2c^\xi = c^{x_1} + c^{x_2}. \quad (16)$$

If (x_1) is the younger of the lives, then let $x_2 = x_1 + a$, and we have

$$2c^\xi = c^{x_1}(1 + c^a). \quad (17)$$

From (17) we have

$$\xi = x_1 + k,$$

where k depends only upon the difference of the ages a and not on the actual ages x_1 and x_2 . Hence, for a given difference between ages, we add the same number to one age to get the equivalent equal ages no matter what the ages x_1 and x_2 are,

To look at this in the general case of m lives we may note that if

$$mc^\xi = c^{x_1} + c^{x_2} + \cdots + c^{x_m}, \quad (18)$$

then

$$mc^{\xi+t} = c^{x_1+t} + c^{x_2+t} + \cdots + c^{x_m+t} \quad (19)$$

for t years later, where t is any number. Thus, if we have obtained the equivalent equal age ξ at any date, then $\xi + t$ is the equivalent equal age for a date t years later. That is, the addition of a given number of years to each of the ages $x_1, x_2, \cdots, x_m$ will add the same number to ξ . This property of Gompertz's and Makeham's functions is known as uniform seniority. The applications of Makeham's formula are much simplified because of this property.

5. A single life may replace m lives by an appropriate change of the interest rate. For certain purposes, it is desirable to work with a single life ξ for which the cost of an annuity would be the same as for joint lives (x_1), (x_2), $\cdots$, (x_m). When Gompertz's law holds, the replacement by a single life is possible without involving a change in the interest rate. But with Makeham's law, we proceed as follows.

We confine our attention to two lives of (x) and (y) since this appears simpler than m lives, although the plan can at once be extended to any number of lives.

Recalling that

$${}_t p_x = s^t g^{c^x(c^t-1)}, \quad (20)$$

and

$${}_t p_y = s^t g^{c^y(c^t-1)}, \quad (21)$$

let ξ be the single life to replace (x) and (y) in joint life probabilities. Then

$${}_t p_\xi = s^t g^{c^\xi(c^t-1)}. \quad (22)$$

From (20) and (21),

$${}_t p_{xy} = s^{2t} g^{(c^x+c^y)(c^t-1)} = s^t {}_t p_\xi, \quad (23)$$

where ξ is given by the equation

$$c^\xi = c^x + c^y. \quad (24)$$

But the value of a life annuity is given by

$$a_{xy} = v p_{xy} + v^2 {}_2 p_{xy} + v^3 {}_3 p_{xy} + \cdots + v^t {}_t p_{xy} + \cdots \quad (25)$$

Let $sv = v'$ be the modified discount factor. This indicates the change in interest rate to which we refer above. Then from (23),

$$v^t {}_t p_{xy} = v'^t {}_t p_\xi. \quad (26)$$

Hence

$$a_{xy} = v' p_\xi + v'^2 {}_2 p_\xi + v'^3 {}_3 p_\xi + \cdots, \quad (27)$$

and we obtain the joint life annuities from the single age ξ .

For special cases, and unusual rates of interest, the actuary is likely to find the substitution of a single life more convenient than the substitution of lives of equal age in number equal to the given number of lives. However, when complete tables are available, the plan that does not require a change of interest rate is likely to be preferred.

6. Makeham's second modification of the law of Gompertz. A second modification of the law of Gompertz was developed by Makeham in 1890.¹

It may be recalled that the first modification assumed that the force of mortality

$$\mu_x = A + Bc^x,$$

where A is a constant.

It is merely another step in the direction of generality to assume that μ_x is the sum of the term Bc^x used in the hypothesis of Gompertz, and a linear function $A + Dx$. We thus start from the assumption

$$\mu_x = A + Dx + Bc^x, \quad (28)$$

or

$$-\frac{1}{l_x} \frac{dl_x}{dx} = A + Dx + Bc^x.$$

¹ *Journal of the Institute of Actuaries*, vol. 28, pp. 152-159, pp. 185-192, pp. 316-331.

Then

$$l_x = ke^{-Ax-(Dx^2/2)-B'e^x} = ks^x r^{x^2} g^{e^x}. \quad (29)$$

$$l_{x+t} = ks^{x+t} r^{(x+t)^2} g^{e^{x+t}},$$

$${}_t p_x = s^t r^{2xt+t^2} g^{e^x(c^t-1)},$$

$${}_t p_y = s^t r^{2yt+t^2} g^{e^y(c^t-1)},$$

$${}_t p_{xy} = s^{2t} r^{2t(x+y)+2t^2} g^{(e^x+c^y)(c^t-1)}. \quad (30)$$

Let ξ be the age of equivalent equal lives, where ξ is defined by

$$2c^\xi = c^x + c^y,$$

in case such a replacement by equal ages is possible.

Then

$${}_t p_{\xi\xi} = s^{2t} r^{4t\xi+2t^2} g^{2c^\xi(c^t-1)}. \quad (31)$$

In the calculation of a joint life annuity,

$$a_{xy} = \sum_{t=1}^{t=\omega} v^t {}_t p_{xy}. \quad (32)$$

To determine a discount factor v' such that

$$v^t {}_t p_{xy} = v'^t {}_t p_{\xi\xi}, \quad (33)$$

we simply substitute in (33) from (30) and (31). This gives after some simplification

$$vr^{2(x+y)} = v' r^{4\xi}$$

or

$$v' = vr^{2(x+y)-4\xi}. \quad (34)$$

That is to say, using a discount factor v' given by (34) instead of v , enables us to replace lives of unequal ages by the same number of lives of equal age. Moreover, the law of uniform seniority holds for the same reasons given in other cases.

This second modification has been used to graduate the new American-Canadian Mortality Table for Men, recently prepared at much labor and expense, under the auspices of the Actuarial Society of America and the American Institute of Actuaries.

It seems that further extensions of Makeham's functions expressive of human mortality could be made by using in place of $A + Dx$ in (28) a polynomial of higher degree, but there would soon come a practical limit to the usefulness of such extensions.

In conclusion, let me say that the main purpose in presenting this paper to the Mathematical Association is accomplished if I have succeeded in giving a general notion of those properties of Makeham's functions that make these functions of much practical value in the treatment of joint life and survivorship insurances and annuities.